A note on homological mirror symmetry for singularities of type D

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Abstract

We prove homological mirror symmetry for Lefschetz fibrations obtained as disconnected sums of polynomials of types A or D. The proof is based on the behavior of the Fukaya category under the addition of a polynomial of type D.

1   Introduction

Let n be a positive integer. An invertible $n \times n$-matrix $A = (a_{ij})_{i,j=1}^n$ with integer components defines a polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ by

$$W = \sum_{i=1}^n x_1^{a_{i1}} \cdots x_n^{a_{in}}.$$

Note that non-zero coefficients of W can be absorbed by rescaling x_i. A polynomial obtained in this way is called an *invertible polynomial* if it has an isolated critical point at the origin. The quotient ring $R = \mathbb{C}[x_1, \ldots, x_n]/(W)$ is naturally graded by the abelian group L generated by $n+1$ elements $\vec{x}_i$ and $\vec{c}$ with relations

$$a_{i1}\vec{x}_1 + \cdots + a_{in}\vec{x}_n = \vec{c}, \qquad i = 1, \ldots, n.$$

The *bounded stable derived category* of R introduced by Buchweitz [Buc87] is the quotient category

$$D_{\mathrm{sing}}^b(R) = D^b(\mathrm{gr}R)/D^{\mathrm{perf}}(\mathrm{gr}R)$$

of the bounded derived category $D^b(\mathrm{gr}R)$ of finitely-generated L-graded R-modules by its full subcategory $D^{\mathrm{perf}}(\mathrm{gr}R)$ consisting of bounded complexes of projectives . This category originates from the theory of *matrix factorizations* introduced by Eisenbud [Eis80], and studied by Orlov [Orl04] under the name '*triangulated category of singularities*'. This category is not necessarily closed under direct summands, and its idempotent completion will be denoted by $D_{\mathrm{sing}}^{\pi}(R)$.

The *transpose* of the invertible polynomial W is defined by

$$W^* = \sum_{i=1}^n x_1^{a_{1i}} \cdots x_n^{a_{ni}},$$

which can be perturbed to an exact Lefschetz fibration with respect to the standard Euclidean Kähler form on $\mathbb{C}^n$. Let $\mathfrak{Fuk}\, W^*$ be the directed A_∞-category defined by Seidel

[Sei01, Sei08] whose set of objects is a distinguished basis of vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes.

The following conjecture comes from the combination of transposition mirror symmetry by Berglund and Hübsch [BH93] and homological mirror symmetry by Kontsevich [Kon95]:

Conjecture 1.1. *For an invertible polynomial W, there is an equivalence*

$$D^b_{\mathrm{sing}}(R) \cong D^b \mathfrak{Fuk}\, W^* \tag{1.1}$$

of triangulated categories.

Conjecture 1.1 is known to hold for Brieskorn-Pham singularities [FU]. Takahashi and Ebeling [Tak, ET] studies Conjecture 1.1 from the point of view of the duality of regular systems of weights by Saito [Sai98].

Recall that the polynomials of types A_n and D_n are defined by

$$f = x^{n+1}$$

and

$$f = x^{n-1} + xy^2$$

respectively. We prove the following in this paper:

Theorem 1.2. *One has an equivalence*

$$D^\pi_{\mathrm{sing}}(R) \cong D^b \mathfrak{Fuk}\, W^* \tag{1.2}$$

of triangulated categories if W^* is a disconnected sum of polynomials of types A or D.

The proof is based on the study of the behavior of categories on both sides of (1.2) under the addition of a polynomial of type D.

The organization of this paper is as follows: In Section 2, we compute the Fukaya category of a Lefschetz fibration defined by a polynomial of type D. In Section 3, we use induction to compute the Fukaya category of a disconnected sum of type A and type D polynomials. The bounded stable derived category of the transpose of a type D singularity is computed in Section 4, and the behavior of stable derived categories under disconnected summation of polynomials is studied in Section 5. In Section 6, we discuss a possible generalization of Conjecture 1.1 to the case with group actions when $n = 2$.

2   The Fukaya category of $x^{n-1} + xy^2$

Let

$$f(x, y) = x^{n-1} + xy^2 + y$$

be a perturbation of a polynomial of type D_n. The critical points of f are given by

$$\begin{cases} x^n = -\dfrac{1}{4(n-1)}, \\ y = -\dfrac{1}{2x}, \end{cases}$$

with critical values
$$f(x, y) = -\frac{n}{n-1} \cdot \frac{1}{4x}.$$

Consider the diagram

$$\Psi = \psi \circ \varpi$$

$$\mathbb{C}^2 \xrightarrow{\varpi} \mathbb{C}^2 \xrightarrow{\psi} \mathbb{C}$$

where
$$\varpi(x, y) = (f(x, y), x)$$

and
$$\psi(y_1, y_2) = y_1.$$

For general $t \in \mathbb{C}$, the map
$$\mathcal{E}_t \xrightarrow{\varpi_t} \mathcal{S}_t$$

from $\mathcal{E}_t = \Psi^{-1}(t)$ to $\mathcal{S}_t = \psi^{-1}(t)$ is a double cover branching at

$$\{x \in \mathbb{C} \mid 4x^n - 4tx - 1 = 0\}.$$

Besides these branch points, the origin is a distinguished point with respect to the projection ϖ_t, since one of two points in the fiber $\varpi_t^{-1}(x)$ goes to infinity at $x = 0$.

Now choose a distinguished set of vanishing paths as the straight line segments from the origin to critical values as shown in Figure 2.1. The trajectories of the branch points of ϖ_t along these paths are shown in Figure 2.2, which are the images of the vanishing cycles by ϖ_0. By the mutation of the distinguished set of vanishing paths as in Figure 2.3, one obtains the vanishing cycles shown in Figure 2.4. By continuing mutations, one arrives at the distinguished set of vanishing paths shown in Figure 2.5. The images of the corresponding vanishing cycles by ϖ_0 are shown in Figure 2.6.

Note that one can perturb C_1 by a Hamiltonian diffeomorphism $\psi : f^{-1}(0) \to f^{-1}(0)$ so that $\psi(C_1)$ does not intersect with C_2. Now it is easy to see that there is a quasi-equivalence

$$D^b \mathfrak{Fuk} f \xrightarrow{\sim} D^b \bmod \Gamma$$

from the derived Fukaya category of f to the derived category of finite-dimensional modules over the path algebra of the quiver

$$\Gamma = \left(\begin{array}{c} v_1 \\ \diagdown \\ \\ \diagup \\ v_2 \end{array} \; v_3 \longleftarrow v_4 \longleftarrow \cdots \longleftarrow v_n \right) \tag{2.1}$$

such that the vanishing cycle C_i is mapped to the simple module S_i associated with the i-th vertex v_i for $i = 1, \ldots, n$.
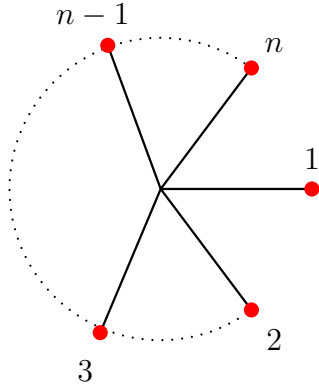
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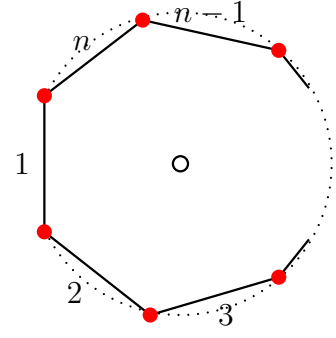Figure 2.1: A distinguished set of vanishing paths

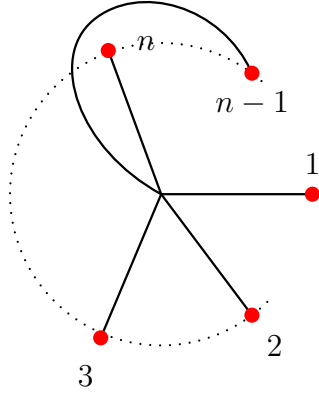Figure 2.2: The image of the vanishing cycles
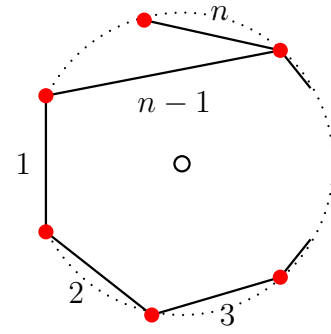

Figure 2.3: Vanishing paths after a mutation
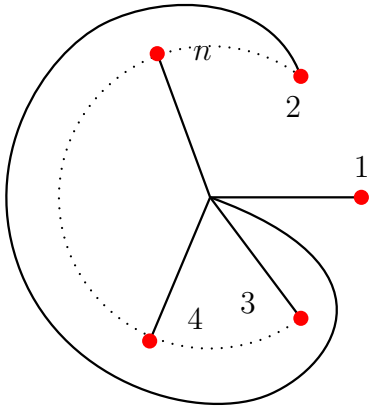

Figure 2.4: Corresponding vanishing cycles

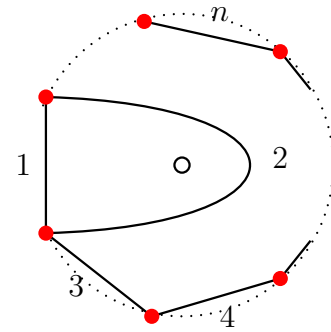

Figure 2.5: Vanishing paths after a sequence of mutations

Figure 2.6: Corresponding vanishing cycles

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3 The Fukaya category of $x^{n-1} + xy^2 + g(z)$

Let

$$f(x,y) = x^{n-1} + xy^2 + y$$

be a perturbation of a polynomial of type D_n and

$$g : \mathbb{C}^k \to \mathbb{C}$$

be an exact symplectic Lefschetz fibration. Consider the diagram

$$\Psi = \psi \circ \varpi$$
$$\mathbb{C}^{k+2} \xrightarrow{\varpi} \mathbb{C}^2 \xrightarrow{\psi} \mathbb{C}$$

where

$$\varpi(x,y,z) = (f(x,y) + g(z), x)$$

and

$$\psi(y_1, y_2) = y_1.$$

We write the critical points of f and g as

$$\operatorname{Crit} f = \{(x_i, y_i)\}_{i=1}^n$$

and

$$\operatorname{Crit} g = \{z_j\}_{j=1}^m,$$

so that the set of critical points of Ψ is given by

$$\operatorname{Crit} \Psi = \operatorname{Crit} f \times \operatorname{Crit} g = \{p_{ij} = (x_i, y_i, z_j)\}_{i,j}$$

with critical values

$$\Psi(p_{ij}) = f(x_i, y_i) + g(z_j).$$

Assume for simplicity that the set of critical values of g is the set of m-th roots of unity. Figure 3.1 shows the critical values of f and g in the case $n = 4$ and $m = 3$, and Figure 3.2 shows the corresponding critical values of Ψ.

For general $t \in \mathbb{C}$, the map

$$\mathcal{E}_t \xrightarrow{\varpi_t} \mathcal{S}_t$$

from $\mathcal{E}_t = \Psi^{-1}(t)$ to $\mathcal{S}_t = \psi^{-1}(t)$ is a Lefschetz fibration away from the origin, and a point $(t, x) \in \mathcal{S}_t$ is a critical value of ϖ_t if there is a solution to the equations

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = \dfrac{\partial g}{\partial z}(z) = 0, \\ f(x,y,z) = t. \end{cases}$$

This condition is equivalent to

$$\begin{cases} 4x^n - 4sx - 1 = 0, \text{ and} \\ s = t - g(z_j) \text{ for some } j \in \{1, \dots, m\}. \end{cases}$$
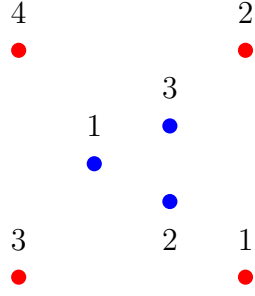
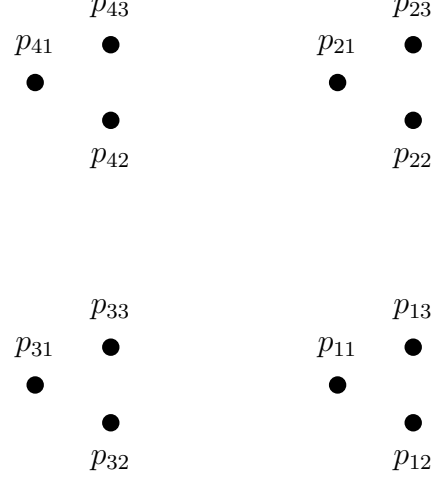Figure 3.1: Critical values of f (outside) and g (inside)

Figure 3.2: Critical values of Ψ

If we write

$$D(s) = \{x \in \mathbb{C} \mid 4x^n - 4sx - 1 = 0\},$$

then the trajectory of $D(t - \mathrm{Crit}(g))$ as t varies along a vanishing path is a matching path corresponding to a vanishing cycle of Ψ. We choose a distinguished set $(\gamma_{ij})_{ij}$ of vanishing paths as in Figure 3.3. The trajectories of $t - \mathrm{Crit}(g)$ are shown in Figure 3.4, and the matching paths μ_{ij} corresponding to vanishing cycles C_{ij} are shown in Figure 3.5. Figure 3.6 is obtained from Figure 3.5 by distorting for better legibility.

Let $C_j \subset g^{-1}(0)$ be the vanishing cycle of g along the straight line segment from the origin to $g(z_j)$, and choose a base point $*$ on the x-plane and a distinguished set (δ_{ij}) of vanishing paths for ϖ_0 as in Figure 3.7. Since $f(x, y)$ is quadratic in the variable y, the fiber $\varpi_0^{-1}(*)$ is a suspension of $g^{-1}(0)$, and the vanishing cycle $\Delta_{i,j}$ of ϖ_0 along δ_{ij} is a suspension of C_j. If write the Fukaya category of $g^{-1}(0)$ consisting of $\{C_j\}_{j=1}^m$ and its directed subcategory as $\mathcal{B}$ and $\mathcal{A}$ respectively, then the Fukaya category $\mathcal{B}_n^\sigma$ of $\varpi_0^{-1}(*)$ consisting of $\Delta_{i,j}$ satisfies

$$\hom_{\mathcal{B}_n^\sigma}(\Delta_{i,j}, \Delta_{i',j'}) = \hom_{\mathcal{B}^\sigma}(C_j^\sigma, C_{j'}^\sigma),$$

where $(\mathcal{B}^\sigma, \mathcal{A}^\sigma)$ is the algebraic suspension of the pair $(\mathcal{B}, \mathcal{A})$ defined by Seidel [Sei]. Let

$$S_{ij} = \mathrm{Cone}(\Delta_{i+1,j} \to \Delta_{i,j}), \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

be the object of $D^b(\mathcal{A}_n^\sigma)$ defined as the cone over the morphism $e_{i,j} : \Delta_{i+1,j} \to \Delta_{i,j}$ corresponding to $\mathrm{id}_{C_j^\sigma}$ under the isomorphism

$$\hom_{\mathcal{A}_n^\sigma}(\Delta_{i+1,j}, \Delta_{i,j}) = \hom_{\mathcal{B}^\sigma}(C_j^\sigma, C_j^\sigma).$$

The following theorem gives a description of the A_∞-structure on C_{ij} for $i \neq 2$ in terms of the Fukaya category of $g^{-1}(0)$:

Theorem 3.1 (Seidel [Sei08, Proposition 18.21])**.** *There is a cohomologically full and faithful functor from the Fukaya category of $\Psi^{-1}(0)$ consisting of vanishing cycles C_{ij} for $i \neq 2$ to $D^b \mathcal{A}_n^\sigma$, such that C_{1j} are mapped to S_{1j} and C_{ij} for $i \geq 3$ are mapped to $S_{i-1,j}$.*

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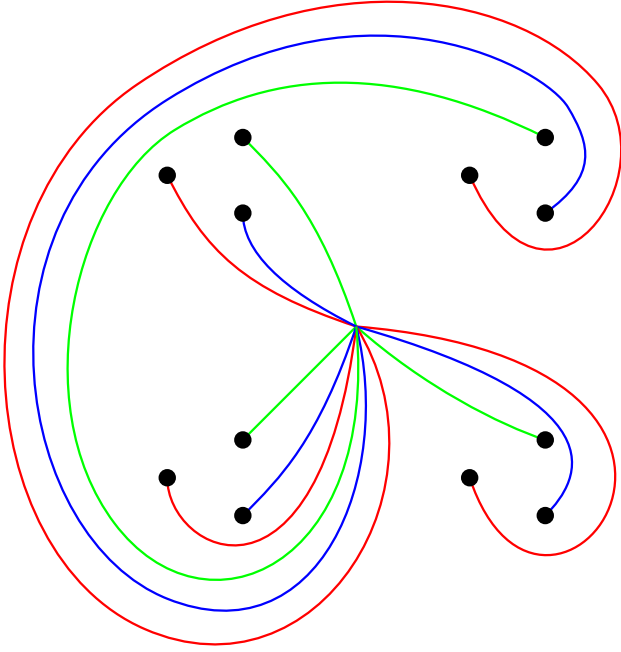Figure 3.4: Trajectories of $t - \mathrm{Crit}(g)$ along the vanishing paths γ_{ij}

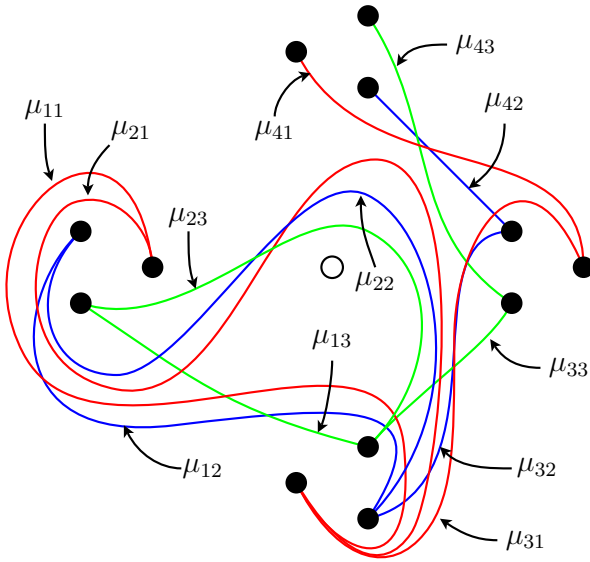Figure 3.3: A distinguished set $(\gamma_{ij})_{ij}$ of vanishing paths for Ψ
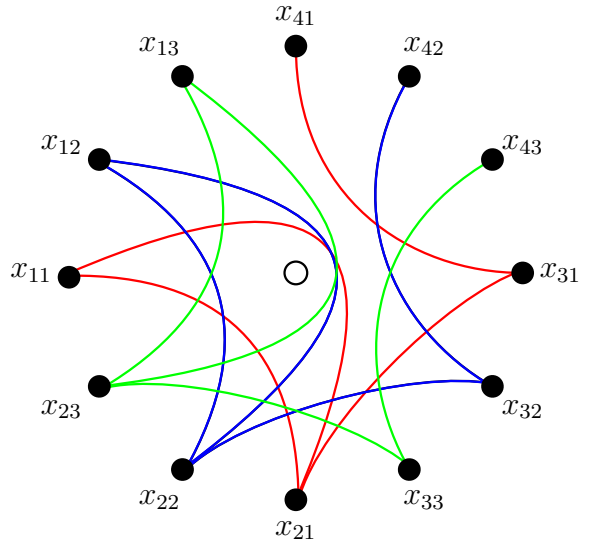

Figure 3.5: Matching paths on the x-plane

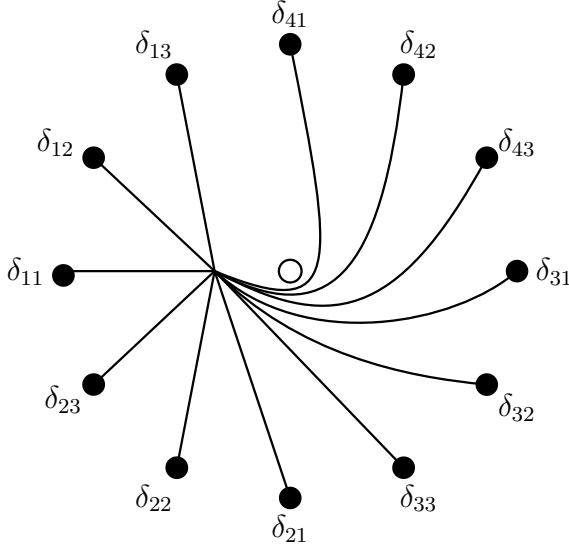Figure 3.6: Matching paths after distortion

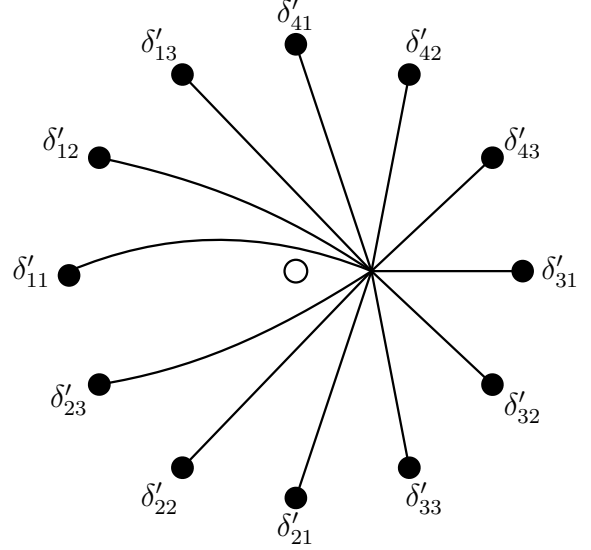Figure 3.7: A distinguished set $(\delta_{ij})_{ij}$ of van-
ishing paths for ϖ_0

Figure 3.8: Another distinguished set $(\delta'_{ij})_{ij}$
of vanishing paths for ϖ_0

Note that this description is completely parallel to the case of type A singularities
given in [FU]. One can repeat the same argument using the distinguished set of vanishing
paths in Figure 3.8 to give an identical description of the A_∞-structure on C_{ij} for $i \neq 1$.
Moreover, since the vanishing cycles C_{1j} and C_{2j} can be obtained as iterated suspensions of
the vanishing cycles C_1 and C_2 in Section 2, one can choose a Hamiltonian diffeomorphism
$\psi : \Psi^{-1}(0) \to \Psi^{-1}(0)$ satisfying $\psi(C_{1j}) \cap C_{2j'} = \emptyset$ for any j and j'. This shows that there
are no morphisms from C_{1j} to $C_{2j'}$ in the cohomology category of the Fukaya category of
Ψ.

Now one can follow the same argument as [FU, Section 3] to show that the Fukaya
category of a disconnected sum of polynomials of types A and D is quasi-equivalent to
the tensor product of the graded categories associated with individual polynomials; it
is straightforward to check this at the level of cohomology categories, and higher A_∞-
operations vanish for degree reasons.

4   The stable derived category of $x^{n-1}y + y^2$

Let
$$W = x^{n-1}y + y^2.$$
be the transpose of a polynomial of type D_n. The abelian group
$$L = \mathbb{Z}\vec{x} \oplus \mathbb{Z}\vec{y} \oplus \mathbb{Z}\vec{c}/((n-1)\vec{x} + \vec{y} - \vec{c}, 2\vec{y} - \vec{c})$$
is isomorphic to $\mathbb{Z}$, so that the coordinate ring
$$R = \mathbb{C}[x, y]/(W)$$
is graded as
$$\deg(x, y) = (1, n - 1).$$

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First recall the following:

Lemma 4.1 (Orlov [Orl05, Lemma 2.4]). *There is a weak semiorthogonal decomposition*

$$D^b(\mathrm{gr}R_{\geq 0}) = \langle \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle$$

where $\mathrm{gr}R_{\geq 0}$ *is the abelian category of* $\mathbb{N}$*-graded* R*-modules,* $\mathcal{P}_{\geq 0}$ *is the subcategory of* $D^b(\mathrm{gr}R_{\geq 0})$ *generated by* $R(i)$ *for* $i \leq 0$, *and* $\mathcal{T}_0$ *is equivalent to* $D^b_{\mathrm{sing}}(R)$.

Let $\mathfrak{m} = (x, y)$ be the maximal ideal of the origin. For a graded R-module M, another graded R-module obtained by shifting the degree of M by $a \in \mathbb{Z}$ will be denoted by $M(a)$; $M(a)_\ell = M_{a+\ell}$.

Lemma 4.2. *The graded modules* $R/(y)$, $R/(x^{n-1}+y)$ *and* $R/\mathfrak{m}(i)$ *for* $i = -n+3, -n+4, \ldots, -1, 0$ *belong to* $\mathcal{T}_0$.

Proof. The projective resolutions

$$\cdots \to \begin{matrix} R(-2n+1) \\ \oplus \\ R(-3n+3) \end{matrix} \xrightarrow{\begin{pmatrix} -y & x^{n-2}y \\ x & y \end{pmatrix}} \begin{matrix} R(-n) \\ \oplus \\ R(-2n+2) \end{matrix} \xrightarrow{\begin{pmatrix} -y & x^{n-2}y \\ x & y \end{pmatrix}} \begin{matrix} R(-1) \\ \oplus \\ R(-n+1) \end{matrix}$$

$$\xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to R/\mathfrak{m} \to 0,$$

$$\cdots \to R(-2n+2) \xrightarrow{x^{n-1}+y} R(-n+1) \xrightarrow{y} R \to R/(y) \to 0,$$

$$\cdots \to R(-2n+2) \xrightarrow{y} R(-n+1) \xrightarrow{x^{n-1}+y} R \to R/(x^{n-1}+y) \to 0$$

show that

$$\mathbb{R}\mathrm{Hom}_{D^b \bmod R}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}(-n+2)[-1],$$
$$\mathbb{R}\mathrm{Hom}_{D^b \bmod R}(R/(y), R) \cong R/(x^{n-1}+y)(-n+1),$$
$$\mathbb{R}\mathrm{Hom}_{D^b \bmod R}(R/(x^{n-1}+y), R) \cong R/(y)(-n+1).$$

It follows that

$$\mathbb{R}\mathrm{Hom}_{D^b\mathrm{gr}R}(R/(y), R(\ell)) = 0, \qquad \ell \leq 0,$$
$$\mathbb{R}\mathrm{Hom}_{D^b\mathrm{gr}R}(R/(x^{n-1}+y), R(\ell)) = 0, \qquad \ell \leq 0,$$
$$\mathbb{R}\mathrm{Hom}_{D^b\mathrm{gr}R}(R/\mathfrak{m}(i), R(\ell)) = 0, \qquad \ell \leq 0,$$

for $i = -n+3, -n+4, \ldots, -1, 0$, and the lemma is proved. $\qquad\square$

Now we have the following:

Lemma 4.3. *The graded modules* $R/(y)$, $R/(x^{n-1}+y)$ *and* $R/\mathfrak{m}(i)$ *for* $i = -n+3, -n+4, \ldots, -1, 0$ *generate* $D^b_{\mathrm{sing}}(R)$ *as a triangulated category.*

Proof. The exact sequence

$$0 \to R \to R/(y) \oplus R/(x^{n-1}+y) \to R/(x^{n-1},y) \to 0$$

shows that the module $R/(x^{n-1},y)$ can be obtained from $R/(y)$ and $R/(x^{n-1}+y)$ by taking cones up to perfect complexes. Then the exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & R/\mathfrak{m}(-1) & \to & R/(x^2,y) & \to & R/\mathfrak{m} & \to & 0, \\
0 & \to & R/\mathfrak{m}(-2) & \to & R/(x^3,y) & \to & R/(x^2,y) & \to & 0, \\
& & & & \vdots & & & & \\
0 & \to & R/\mathfrak{m}(-n+2) & \to & R/(x^{n-1},y) & \to & R/(x^{n-2},y) & \to & 0,
\end{array}
$$

show that $R/\mathfrak{m}(-n+2)$ can be obtained from $R/(y)$, $R/(x^{n-1}+y)$, $R/\mathfrak{m}$, ..., $R/\mathfrak{m}(-n+2)$, and $R/\mathfrak{m}(-n+3)$ by taking cones up to perfect complexes. Now note that

$$0 \to R/(y)(-1) \to R/(y) \to R/\mathfrak{m} \to 0$$

shows that $R/(y)(-1)$ can be obtained from $R/(y)$ and $R/\mathfrak{m}$, and $R/(x^{n-1}+y)(-1)$ can be obtained from $R/(x^{n-1}+y)$ and $R/\mathfrak{m}$ in the same way. Then by repeating the above argument with degree shifted, one can obtain $R/\mathfrak{m}(-n+1)$, $R/\mathfrak{m}(-n)$ and so on.

Now consider the exact sequences

$$0 \to R(-1) \xrightarrow{x} R \to R/(x,y^2)$$

and

$$0 \to R/\mathfrak{m}(-n+1) \to R/(x,y^2) \to R/\mathfrak{m} \to 0.$$

These show that $R/\mathfrak{m}(a+n-1)$ can be obtained from $R/\mathfrak{m}(a)$ by taking cones up to perfect complexes.

Alternatively, one can use a result of Orlov [Orl05, Theorem 2.5] and the fact that R is Gorenstein with the parameter $2-n$, which gives the weak semiorthogonal decomposition

$$\mathcal{T}_0 = \langle R/\mathfrak{m}, R/\mathfrak{m}(-1), \ldots, R/\mathfrak{m}(-n+3), \mathcal{D}_{n-2} \rangle$$

where $\mathcal{T}_0$ is equivalent to $D^b_{\mathrm{sing}}(R)$ and $\mathcal{D}_{n-2}$ is equivalent to $\mathrm{qgr}(R)$. Although $R/(y)$ and $R/(x^{n-1}+y)$ do not belong to $\mathcal{D}_{n-2}$, suitable cones over $R/(y)$, $R/(x^{n-1}+y)$ and M_i for $i = -n+3, \ldots, 0$ belongs to $\mathcal{D}_{n-2}$, and these cones generate $\mathcal{D}_{n-2} \cong \mathrm{qgr}(R)$. $\square$

By computing the Ext-groups between these generators, one can show the equivalence

$$D^b_{\mathrm{sing}}(R) \cong D^b \bmod \Gamma$$

with the derived category of finite-dimensional representations of the Dynkin quiver Γ in (2.1), such that the graded modules $R/(y)$, $R/(x^{n-1}+y)$ and $R/\mathfrak{m}(i)$ for $i = 0, -1, \ldots, -n+3$ goes to the simple modules corresponding to the vertices v_1, v_2 and v_{-i+3}.

5 The stable derived category of a disconnected sum

We discuss a graded analogue of a result of Dyckerhoff [Dyc, Theorem 5.1] in this section. Let W be an invertible polynomial in $S = \mathbb{C}[x_1, \ldots, x_n]$ graded by L. A *graded matrix factorization* is an infinite sequence

$$K^\bullet = \{\cdots \to K^i \xrightarrow{k^i} K^{i+1} \xrightarrow{k^{i+1}} K^{i+2} \to \cdots\}$$

of morphisms of L-graded free S-modules such that $k^{i+1} \circ k^i = W$ and $K^\bullet[2] = K^\bullet(\vec{c})$. A morphism between two graded matrix factorizations $K^\bullet$ and $H^\bullet$ is a family of morphisms $f^i : K^i \to H^i$ of L-graded modules such that $f^{i+2} = f^i(\vec{c})$. The composition and the differential on morphisms are defined in just the same way as unbounded complexes of L-graded modules. The homotopy category of the differential graded category $\mathrm{mf}_S^L(W)$ of finitely-generated graded matrix factorizations is equivalent to the stable category of graded maximal Cohen-Macaulay modules over $R = S/(W)$ by Eisenbud [Eis80], which in turn is equivalent to the bounded stable derived category $D_{\mathrm{sing}}^b(R)$ by Buchweitz [Buc87].

Recall that an object $\mathcal{E}$ in a triangulated category is a *generator* if $\mathrm{Hom}(\mathcal{E}, \mathcal{F}[i]) = 0$ for all $i \in \mathbb{Z}$ implies $\mathcal{F} \cong 0$. Let $\overline{L} \subset L$ be a representative of the finite abelian group $L/\mathbb{Z}\vec{c}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal of S corresponding to the origin. Since R has an isolated singularity at the origin, a result of Schoutens [Sch03], Murfet [KMVdB, Proposition A.2], Orlov [Orl09, Proposition 2.7], or Dyckerhoff [Dyc, Corollary 4.3] shows that the object

$$\mathcal{E} = \bigoplus_{\ell \in \overline{L}} R/\mathfrak{m}(\ell)$$

is a generator of $D_{\mathrm{sing}}^b(R)$.

The graded matrix factorization $K^\bullet$ corresponding to $R/\mathfrak{m}$ is given by

$$K^i = \begin{cases} \displaystyle\bigoplus_{j\,:\,\mathrm{even}} \Omega_S^j & i \text{ is even,} \\ \displaystyle\bigoplus_{j\,:\,\mathrm{odd}} \Omega_S^j & i \text{ is odd,} \end{cases}$$
$$k^i = \iota_\eta + \gamma \wedge \cdot$$

with a suitable grading, where

$$\eta = \sum_i x_i \partial_i$$

is the Euler vector field and γ is a one-form such that

$$W = \gamma(\eta).$$

Now assume that W is a disconnected sum of two invertible polynomials $W_1 \in S_1$ and $W_2 \in S_2$, and $\overline{L}$ is the image of the product $\overline{L}_1 \times \overline{L}_2$ of representatives $\overline{L}_1 \subset L_1$ and $\overline{L}_2 \subset L_2$ by the natural projection

$$L_1 \oplus L_2 \to L = L_1 \oplus L_2/(\vec{c}_1 - \vec{c}_2).$$

Then the matrix factorization $K^\bullet$ for W is the tensor product of matrix factorizations $K_1^\bullet$ and $K_2^\bullet$ for W_1 and W_2;

$$K^i = \begin{cases} \Omega_{S_1}^{\text{even}} \otimes \Omega_{S_2}^{\text{even}} \oplus \Omega_{S_1}^{\text{odd}} \otimes \Omega_{S_2}^{\text{odd}} & i \text{ is even,} \\ \Omega_{S_1}^{\text{even}} \otimes \Omega_{S_2}^{\text{odd}} \oplus \Omega_{S_1}^{\text{odd}} \otimes \Omega_{S_2}^{\text{even}} & i \text{ is odd,} \end{cases}$$

$$k^i = (\iota_{\eta_1} + \gamma_1 \wedge \cdot) + (\iota_{\eta_2} + \gamma_2 \wedge \cdot).$$

It follows that one has a quasi-isomorphism of differential graded algebras

$$\hom_{\text{mod } S}(K^\bullet, K^\bullet) \cong \hom_{\text{mod } S_1}(K_1^\bullet, K_1^\bullet) \otimes \hom_{\text{mod } S_2}(K_2^\bullet, K_2^\bullet),$$

which induces a quasi-isomorphism

$$\text{End}_{\text{mf}_S^L(W)}(\mathcal{E}) \cong \text{End}_{\text{mf}_{S_1}^{L_1}(W_1)}(\mathcal{E}_1) \otimes \text{End}_{\text{mf}_{S_2}^{L_2}(W_2)}(\mathcal{E}_2)$$

by passing to L-graded pieces. Since $\mathcal{E}$ is a compact generator of $\text{mf}_S^L(W)$, this shows that the tensor product $\text{mf}_{S_1}^{L_1}(W_1) \otimes \text{mf}_{S_2}^{L_2}(W_2)$ of differential graded categories is quasi-equivalent to $\text{mf}_S^L(W)$ up to direct summands.

6 Group actions and crepant resolutions

Let W be an invertible polynomial associated with an $n \times n$ matrix $A = (a_{ij})_{ij}$. The abelian group L is the group of characters of K defined by

$$K = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{n1}} \cdots \alpha_n^{a_{nn}}\}.$$

The group G_{max} of *maximal diagonal symmetries* is defined as the kernel of the map

$$\begin{array}{ccc} K & \rightarrow & \mathbb{C}^\times \\ \cup & & \cup \\ (\alpha_1, \ldots, \alpha_n) & \mapsto & \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}}, \end{array}$$

so that there is an exact sequence

$$1 \rightarrow G_{\text{max}} \rightarrow K \rightarrow \mathbb{C}^\times \rightarrow 1.$$

This exact sequence induces an exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow L \rightarrow G_{\text{max}}^\vee \rightarrow 1$$

of the corresponding character groups, where

$$G_{\text{max}}^\vee = \text{Hom}(G_{\text{max}}, \mathbb{C}^\times)$$

is non-canonically isomorphic to G_{max}. If we write

$$A^{-1} = \begin{pmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(n)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n^{(1)} & \varphi_n^{(2)} & \cdots & \varphi_n^{(n)} \end{pmatrix},$$

then the group $G_{\max}$ is generated by

$$\rho_k = \left(\exp\left(2\pi\sqrt{-1}\varphi_1^{(k)}\right), \ldots, \exp\left(2\pi\sqrt{-1}\varphi_n^{(k)}\right)\right), \qquad k = 1, \ldots, n.$$

Put

$$\varphi_i = \varphi_i^{(1)} + \cdots + \varphi_i^{(n)}, \qquad i = 1, \ldots, n$$

and define a homomorphism

$$\varphi : \mathbb{C}^\times \to K$$

by

$$\varphi(\alpha) = \left(\alpha^{\ell\varphi_1}, \ldots, \alpha^{\ell\varphi_n}\right),$$

where ℓ is the smallest integer such that $\ell\varphi_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Then φ is injective and one has an exact sequence

$$1 \to \mathbb{C}^\times \xrightarrow{\varphi} K \to \overline{G}_{\max} \to 1,$$

where $\overline{G}_{\max} := \operatorname{coker}\varphi$. W is quasi-homogeneous of degree ℓ with respect to the $\mathbb{Z}$-grading

$$\deg x_i = \ell\varphi_i, \qquad i = 1, \ldots, n.$$

The intersection $\operatorname{Im}\varphi \cap G$ is generated by

$$J = \left(\exp\left(2\pi\sqrt{-1}\varphi_1\right), \ldots, \exp\left(2\pi\sqrt{-1}\varphi_n\right)\right).$$

Let G be a subgroup of $G_{\max}$ containing J and $\overline{G} = G/\langle J \rangle$ be its image in $\overline{G}_{\max}$. The inverse image of $\overline{G}$ by $K \to \overline{G}_{\max}$ will be denoted by H, and we write the group of characters of H as M. Let A^* be the transpose matrix of A. The group $G_{\max}^*$ of maximal diagonal symmetries of W^* is generated by the column vectors $\overline{\rho}_i$ of $(A^*)^{-1} = (A^{-1})^*$. The *transpose* G^* of the subgroup $G \subset G_{\max}$ is defined by Krawitz [Kra] as

$$G^* = \left\{ \prod_{i=1}^n \overline{\rho}_i^{r_i} \ \middle|\ \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix} A^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z} \text{ for all } \prod_{i=1}^n \rho_i^{a_i} \in G \right\}.$$

The *transposition mirror symmetry* of Berglund and Hübsch [BH93] states that the pairs (W, G) and (W^*, G^*) are mirror dual to each other. Homological mirror symmetry is expected to take the form

$$D_{\mathrm{sing}}^b(R) \cong D^b \mathfrak{Fuk}^{G^*} W^*$$

where $R = \mathbb{C}[x_1, \ldots, x_n]/(W)$ is an M-graded ring, although the "orbifold Fukaya category" $\mathfrak{Fuk}^{G^*} W^*$ on the right hand side is not defined yet.

If W^* is a polynomial in two variables and $G^* \subset SL_2(\mathbb{C})$, then the map

$$W^* : \mathbb{C}^2 \to \mathbb{C}$$

descends to the map

$$\overline{W}^* : \mathbb{C}^2/G^* \to \mathbb{C},$$

which can be pulled-back to the crepant resolution $\pi : Y \to \mathbb{C}^2/G^*$;

$$\widetilde{W}^* = \overline{W}^* \circ \pi : Y \to \mathbb{C}.$$

One can replace $\mathfrak{Fuk}^{G^*} W^*$ with the Fukaya category $\mathfrak{Fuk}\,\widetilde{W}^*$ of a perturbation of $\widetilde{W}^*$ and formulate the following conjecture:

Conjecture 6.1. *If $n = 2$ and $G^* \subset SL_2(\mathbb{C})$, then one has an equivalence*

$$D^b_{\mathrm{sing}}(R) \cong D^b \mathfrak{Fut} \, \widetilde{W}^*$$

of triangulated categories.

As an example, consider the case when

$$W^*(u, v) = u^3 v + v^2$$

is the transpose of a polynomial of type D_4 and $G^* = \left\langle \frac{1}{2}(1, 1) \right\rangle \subset SL_2(\mathbb{C})$ is a cyclic group of order two generated by $\mathrm{diag}(-1, -1)$. The invariant ring $\mathbb{C}^2[x, y]^{G^*}$ is generated by

$$x = u^2,$$
$$y = v^2,$$
$$z = uv$$

with the relation

$$xy = z^2,$$

and $\overline{W}^*$ is given by

$$\overline{W}^* = xz + y.$$

The minimal resolution of

$$S = \mathbb{C}^2/G = \mathrm{Spec} \, \mathbb{C}[x, y, z]/(xy - z^2)$$

is obtained by a blow-up along the ideal $(x, z) \subset \mathbb{C}[x, y, z]$, which is covered by a chart

$$x = x_1,$$
$$y = y,$$
$$z = x_1 z_1,$$

with a local coordinate (x_1, z_1), and another chart

$$x = x_2 z_2,$$
$$y = y,$$
$$z = z_2,$$

with a local coordinate (x_2, y). The map $\widetilde{W}^*$ is written as

$$\widetilde{W}^* = x_1^2 z_1 + x_1 z_1^2$$

on the first chart where it has a D_4-singularity at the origin, and as

$$\widetilde{W}^* = x_2^3 y^2 + y$$

on the second chart, where it does not have any critical point.

The transpose of $\left(W^*, \left\langle \frac{1}{2}(1, 1) \right\rangle \right)$ is given by $\left(x^3 + xy^2, \left\langle \frac{1}{3}(1, 1) \right\rangle \right)$, where $\left\langle \frac{1}{3}(1, 1) \right\rangle \subset GL_2(\mathbb{C})$ is the cyclic group of order three generated by $\mathrm{diag}(\exp(2\pi\sqrt{-1}/3), \exp(2\pi\sqrt{-1}/3))$.

The corresponding abelian group M is isomorphic to $\mathbb{Z}$, and the resulting grading of $R = \mathbb{C}[x, y]/(x^3 + xy^2)$ is given by $\deg x = \deg y = 1$. One can show an equivalence

$$D^b_{\mathrm{sing}}(R) \cong D^b \bmod \Gamma$$

with the derived category of a Dynkin quiver Γ of type D_4 just as in Section 4, and Conjecture 6.1 holds in this case.

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